
Almost Global Convergence in Singular Perturbations of Strongly Monotone Systems

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Abstract. This paper deals with global convergence to equilibria, and in particular Hirsch’s generic convergence theorem for strongly monotone systems, for singular perturbations of monotone systems.

1 Introduction

Monotone systems constitute a rich class of models for which global and almost-global convergence properties can be established. They are particularly useful in biochemical models (see discussion and references in [14, 15]), and also appear in areas like coordination ([11]) and other problems in control theory ([1]). This paper studies extensions, using geometric singular perturbation theory, of Hirsch’s generic convergence theorem for monotone systems ([4, 5, 6, 13]). Informally stated, Hirsch’s result says that almost every bounded solution of a strongly monotone system converges to the set of equilibria. There is a rich literature regarding the application of this powerful theorem, as well as of other results dealing with everywhere convergence when equilibria are unique ([13, 2, 7]), to models of biochemical systems. Unfortunately, many models in biology are not monotone. In order to address this drawback (as well as to study properties of large systems which are monotone but which are hard to analyze in their entirety), a recent line of work introduced an input/output approach that is based on the analysis of interconnections of monotone systems. For example, the approach allows one to view a *non*-monotone system as a “negative” feedback loop of monotone open-loop systems, thus leading to results on global stability and the emergence of oscillations under transmission delays, and to the construction of relaxation oscillators by slow adaptation rules on feedback gains. See [14, 15] for expositions and many references. The present paper is in the same character.

Our motivation arose from the observation that time-scale separation may also lead to monotonicity. This point of view is of special interest in the

context of biochemical systems; for example, Michaelis Menten kinetics are mathematically justified as singularly perturbed versions of mass action kinetics. A system that is not monotone may become monotone once that fast variables are replaced by their steady-state values. A trivial linear example that illustrates this point is $\dot{x} = -x - y$, $\varepsilon \dot{y} = -y + x$, with $\varepsilon > 0$. This system is not monotone with respect to any orthant cone. On the other hand, for $\varepsilon \ll 1$, the fast variable y tracks x , so the slow dynamics is well-approximated by $\dot{x} = -2x$ (which is strongly monotone, because every scalar system is).

We consider systems $\dot{x} = f(x, y)$, $\varepsilon \dot{y} = g(x, y)$ for which the reduced system $\dot{x} = f(x, h(x))$ is strongly monotone (in fact, a slightly stronger technical condition on derivatives is assumed) and the fast system $\dot{y} = g(x, y)$ has a unique globally asymptotically stable steady state $y = h(x)$ for each x , and satisfies an input to state stability type of property with respect to x . One may expect that the original system inherits global (generic) convergence properties, at least for all $\varepsilon > 0$ small enough, and this is indeed the object of our study. This question may be approached in several ways. One may view $y - h(x)$ as an input to the slow system, and appeal to the theory of asymptotically autonomous systems. Another approach, the one that we develop here, is through geometric invariant manifold theory ([3, 8, 12]). There is a manifold M_ε , invariant for the full dynamics, which attracts all near-enough solutions, with an asymptotic phase property. The system restricted to the invariant manifold M_ε is a regular perturbation of the fast ($\varepsilon=0$) system. As remarked in Theorem 1.2 in Hirsch's early paper [4], a C^1 regular perturbation of a flow with eventually positive derivatives also has generic convergence. So, solutions in the manifold will be generally well-behaved, and asymptotic phase implies that solutions track solutions in M_ε , and hence also converge to equilibria if solutions on M_ε do. A key technical detail is to establish that the tracking solutions also start from the "good" set of initial conditions, for generic solutions of the large system.

For simplicity, we discuss here only the case of cooperative systems (monotonicity with respect to the main orthant), but proofs in the case of general cones are similar and will be discussed in a paper under preparation.

2 Statement of Main Result

We are interested in systems in singularly perturbed form:

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \varepsilon \frac{dy}{dt} &= g(x, y), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $0 < \varepsilon \ll 1$, and f and g are smooth functions. We will present some preliminary results in general, but for our main theorem we will restrict attention to the case when g has the special form $g(x, y) = Ay + h(x)$,

where A is a Hurwitz matrix (all eigenvalues have negative real part) and h is a smooth function. That is, we will specialize to systems of the following form:

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) \\ \varepsilon \frac{dy}{dt} &= Ay + h(x).\end{aligned}\tag{2}$$

(We remark later how our results may be extended to a broader class of systems.) Setting ε to zero, we have:

$$\frac{dx}{dt} = f(x, m_0(x)),\tag{3}$$

where $m_0(x) = -A^{-1}h(x)$. As usual in singular perturbation theory, our goal is to use properties of the limiting system (3) in order to derive conclusions about the full system (2) when $0 < \varepsilon \ll 1$. In this paper, $A \subset B$ means that A is a strict subset of B , while $A \subseteq B$ contains the case of $A = B$. We will assume given three sets K , \tilde{K} , and L which satisfy the following hypotheses (some technical terms are defined later):

- H1** The set \tilde{K} is an n -dimensional C^∞ simply connected compact manifold with boundary.
- H2** The set L is a bounded open subset of \mathbb{R}^m , and

$$M_0 = \{(x, y) \mid y = m_0(x), x \in \tilde{K}\},$$

the graph of m_0 , is contained in $\tilde{K} \times L$.

- H3** The flow $\{\psi_t\}$ of the limiting system (3) has eventually positive derivatives on \tilde{K} .
- H4** The set \tilde{K} is convex, and therefore it is p-convex too.
- H5** For each $\varepsilon > 0$ sufficiently small, the forward trajectory under (2) of each point in $\tilde{D} = \text{Int}\tilde{K} \times L$ is precompact in \tilde{D} .
- H6** The equilibrium set $E_0 = \{x \in \text{Int}\tilde{K} \mid f(x, m_0(x)) = 0\}$ is countable.
- H7** The set $K \subset \text{Int}\tilde{K}$ is compact, and for each $\varepsilon > 0$ sufficiently small, the set $D = K \times L$ is positively invariant.

Note that the equilibria of (2) do not depend on ε , and the ones in \tilde{D} are in 1-1 correspondence with elements of E_0 . The main theorem is:

Theorem 1 *Under assumptions H1-H7, there exists $\varepsilon^* > 0$ such that for each $0 < \varepsilon < \varepsilon^*$, the forward trajectory of (2) starting from almost every point in D converges to some equilibrium.*

Remark: A variant of this result is to assume that the reduced system (3) has a unique equilibrium. In this case, one may improve the conclusions of the theorem to global (not just generic) convergence, by appealing to results of Hirsch and others that apply when equilibria are unique. The proof is simpler in that case, since the foliation structure given by Fenichel's theory (see below) is not required. In the opposite direction, one could drop the assumption of countability and instead provide theorems on generic convergence to the set of equilibria, or even to equilibria if hyperbolicity conditions are satisfied, in the spirit of what is done in the theory of strongly monotone systems. \square

3 Terminology

The following standard terminology is defined for a general ordinary differential equation:

$$\frac{dz}{dt} = F(z), \quad (4)$$

where $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a C^1 vector field. For any $z \in \mathbb{R}^N$, we denote the maximally defined solution of (4) with initial condition z by $t \rightarrow \phi_t(z)$, $t \in I(z)$, where $I(z)$ is an open interval in \mathbb{R} that contains zero. For each $t \in \mathbb{R}$, the set of $z \in \mathbb{R}^N$ for which $\phi_t(z)$ is defined is an open set $W(t) \subseteq \mathbb{R}^N$, and $\phi_t : W(t) \rightarrow W(-t)$ is a diffeomorphism. The collection of maps $\{\phi_t\}_{t \in \mathbb{R}}$ is called the flow of (4). We also write just $z(t)$ for the solution of (4), if the initial condition $z(0)$ is clear from the context. The forward trajectory of $z \in \mathbb{R}^N$ is a parametrized curve $t \rightarrow \phi_t(z)$. Its image is the forward orbit of z , denoted as $O_+(z)$. The backward trajectory and the backward orbit $O_-(z)$ are defined analogously. A set $U \subseteq \mathbb{R}^N$ is *positively (respectively, negatively) invariant* if $O_+(U) \subseteq U$ ($O_-(U) \subseteq U$). It is *invariant* if it is both positively and negatively invariant.

We borrow the notation from [8] for the forward evolution of a set $U \subseteq V \subseteq \mathbb{R}^N$ restricted to V :

$$U \cdot_V t = \{\phi_t(p) : p \in U \text{ and } \phi_s(p) \in V \text{ for all } 0 \leq s \leq t\}.$$

Let us denote the interior and the closure of a set U as $\text{Int}U$ and \overline{U} respectively.

Definition 1 *The flow $\{\phi_t\}$ of (4) is said to have eventually positive derivatives on a set $V \subseteq \mathbb{R}^N$ if there exists t_0 such that $\frac{\partial \phi_t^i}{\partial z_j}(z) > 0$ for all $t \geq t_0$, $z \in V$.*

If (4) is of dimension one, i.e. $N = 1$, $\{\phi_t\}$ has eventually positive derivatives automatically. In practice, the following sufficient condition is easier to check. If the vector field of (4) satisfies $\frac{\partial F_i}{\partial z_j}(z) \geq 0$, for all $z \in V$, $i \neq j$, and the matrix $\frac{\partial F}{\partial z}(z)$ is irreducible for all $z \in V$, then $\{\phi_t\}$ has eventually positive derivatives. (This condition is not necessary.)

Definition 2 *An open set $W \subseteq \mathbb{R}^N$ is called p-convex if W contains the entire line segment joining x and y whenever $x, y \in W$ and $x \leq y$, where $x \leq y$ means $x_i \leq y_i$ for all $i = 1, \dots, N$.*

The next lemma is a restatement of theorem 4.4 in [5]:

Lemma 1. *Suppose that the open set $W \subseteq \mathbb{R}^n$ is p-convex and the flow $\{\phi_t\}$ of (4) has eventually positive derivatives on W . Let $W^c \subseteq W$ be the set of points whose forward orbit has compact closure in W . If the set of equilibrium points is countable, then $z(t)$ converges to a equilibrium as $t \rightarrow \infty$, for almost every $z \in W^c$.*

The following fact follows from differentiability of solutions with respect to “regular” perturbations in the dynamics; see [5], Theorem 1.2:

Lemma 2. *Assume $V \subset W$ is a compact set in which the flow $\{\phi_t\}$ has eventually positive derivatives. Then, there exists $\delta > 0$ with the following property. Let $\{\psi_t\}$ denote the flow of a C^1 vector field G such that the C^1 norm of $F(z) - G(z)$ is less than δ for all z in V . Then there exists $t_* > 0$ such that if $t \geq t_*$ and $\psi_s(z) \in V$ for all $s \in [0, t]$, then $\frac{\partial \psi_t^i}{\partial z^j}(z) > 0$.*

The Appendix reviews the definition of a “ C^r ($1 \leq r \leq \infty$) manifold M with boundary” in the sense used on geometric singular perturbation theory. We denote the boundary of such a manifold M as ∂M , and denote $M \setminus \partial M$ as $\text{Int}M$, when there is no confusion with the notation for the interior of a set.

Definition 3 *A compact, connected C^r manifold $M \subset \mathbb{R}^N$ with boundary is said to be locally invariant under the flow of (1) if for each p in $\text{Int}M$ there exists a time interval $I_p = (t_1, t_2)$, for some $t_1 < 0 < t_2$, such that $\phi_t(p) \in M$ for all $t \in I_p$.*

When $\varepsilon \neq 0$, we can “stretch” the time ($\tau = \frac{t}{\varepsilon}$), and consider the fast system:

$$\begin{aligned}\frac{dx}{d\tau} &= \varepsilon f(x, y) \\ \frac{dy}{d\tau} &= g(x, y).\end{aligned}\tag{5}$$

The above system is equivalent to (1). The corresponding fast system for (2) is

$$\begin{aligned}\frac{dx}{d\tau} &= \varepsilon f(x, y) \\ \frac{dy}{d\tau} &= Ay + h(x).\end{aligned}\tag{6}$$

Definition 4 *Let M be an n -dimensional manifold (possibly with boundary) contained in $\{(x, y) \mid g(x, y) = 0\}$. We say that M is normally hyperbolic relative to (5) if all eigenvalues of the matrix $\frac{\partial g}{\partial y}(p)$ have nonzero real part for every $p \in M$.*

4 Proof of the Main Theorem

Recall the definition of $M_0 = \{(x, y) \mid y = m_0(x), x \in \tilde{K}\}$. Since \tilde{K} is an n -dimensional C^∞ compact manifold with boundary, and m_0 is a smooth function, M_0 is also an n -dimensional C^∞ compact manifold with boundary.

Our proofs are based on Fenichel’s theorems [3], in the forms presented and developed by Jones in [8].

Fenichel’s First Theorem *Under assumption H1, if M_0 is normally hyperbolic relative to (6), then there exists $\varepsilon_0 > 0$, such that for every $0 < \varepsilon < \varepsilon_0$ and $r > 0$, there is a function $y = m_\varepsilon(x)$, defined on \tilde{K} , of class C^r jointly in x and ε , such that*

$$M_\varepsilon = \{(x, y) \mid y = m_\varepsilon(x), x \in \tilde{K}\}$$

is locally invariant under (2), see Figure 1.

The requirement that M_0 be normally hyperbolic is satisfied in our case, as $g(x, y) = Ay + h(x)$ and therefore $\frac{\partial g}{\partial y}(p) = A$, which is Hurwitz, for each $p \in M_0$.

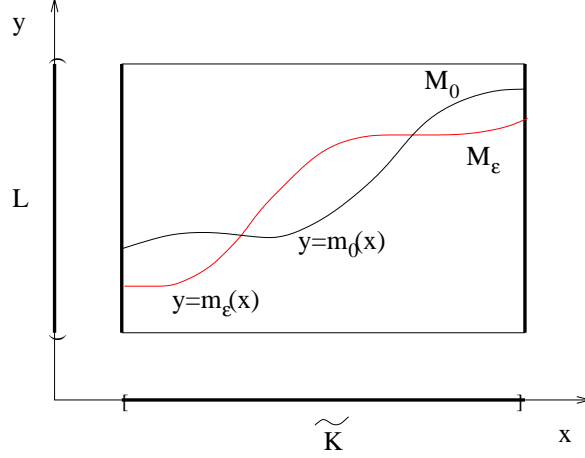


Fig. 1. For simplicity, we sketch manifolds M_ε and M_0 of a system where $n = m = 1$. The set \tilde{K} is a compact set in x , and L is an open set in y . The red curve denotes the locally invariant manifold M_ε and the black curve denotes M_0 .

We will pick a particular $r > 1$ in the above theorem from now on.

Let us interpret local invariance in terms of equations. Let $(x(t), y(t))$ be the solution to (2) with initial condition (x_0, y_0) , such that $x_0 \in \text{Int}\tilde{K}$ and $y_0 = m_\varepsilon(x_0)$. Local invariance implies that $(x(t), y(t))$ satisfies

$$\frac{dx(t)}{dt} = f(x(t), m_\varepsilon(x(t))) \quad (7)$$

$$y(t) = m_\varepsilon(x(t)), \quad (8)$$

for all t small enough. Actually, this is also true for all $t \geq 0$. The argument is as follows. By **H5**, $(x(t), y(t))$ is well-defined and remains in \tilde{D} for all $t \geq 0$. Let $T = \{t \geq 0 \mid y(t) = m_\varepsilon(x(t))\}$. Then, T is not empty, and T is closed by the continuity of $m_\varepsilon(x(t))$ and $y(t)$. Also, T is open, since M_ε is locally invariant. So $T = \{t \geq 0\}$, that is, $x(t)$ is a solution to (7) and $y(t) = m_\varepsilon(x(t))$ for all $t \geq 0$.

In (7), the x -equation is decoupled from the y -equation, which allows us to reduce to studying a lower-dimension system. Another advantage is that, as ε approaches zero, the limit of system (7) is system (3), which describes the flows on M_0 . If M_0 has some desirable property, it is natural to expect

this property is inherited by the perturbed manifold M_ε . An example of this principle is provided by the following lemma.

Lemma 3. *Under assumptions **H1-H3**, for each $0 < \varepsilon < \varepsilon_0$, the flow $\{\psi_t\}$ of (7) has eventually positive derivatives on $\text{Int}\tilde{K}$.*

Proof. Applying Lemma 2, there exist $\delta > 0$ such that when the C^1 norm of $m_0(x) - m_\varepsilon(x)$ is less than δ for all $x \in \tilde{K}$, there exists $t_* > 0$ with the property that: if $t \geq t_*$ and $\psi_s(x) \in \tilde{K}$ for all $s \in [0, t]$, then $\frac{\partial \psi_t^i}{\partial x^j}(x) > 0$. Since m_ε is of class C^r , jointly in x and ε , we can pick $\varepsilon > 0$ small enough to control δ . If we then prove $\text{Int}\tilde{K}$ is invariant under (7), that is, for any $x_0 \in \text{Int}\tilde{K}$, the solution $x(t)$ of (7) with initial condition x_0 stays in $\text{Int}\tilde{K}$ for all $t \geq 0$, then we are done. Let us now prove that $\text{Int}\tilde{K}$ is positively invariant under (7).

Let $y_0 = m_\varepsilon(x_0)$ and $y(t) = m_\varepsilon(x(t))$. Then, $(x(t), y(t))$ is the solution to (2) with initial condition $(x_0, y_0) \in \tilde{D}$. By **H5**, $(x(t), y(t))$ stays in \tilde{D} for all $t \geq 0$, and therefore $x(t) \in \text{Int}\tilde{K}$ for all $t \geq 0$. \square

Flows with eventually positive derivatives have particularly appealing properties, as in Lemma 1. To apply that lemma, we need to check two conditions. First, for every point in $\text{Int}\tilde{K}$, its forward trajectory under (7) has compact closure in $\text{Int}\tilde{K}$. Second, the number of equilibria of (7) is countable. Suppose that the first property does not hold, and let $x(t)$ be a solution to (7) with $x(0) \in \text{Int}\tilde{K}$ but $\lim_{j \rightarrow \infty} x(t_j) \notin \text{Int}\tilde{K}$ for some sequence $\{t_j\}$. So, $(x(t), m_\varepsilon(x(t)))$ is a solution for (2), and its forward orbit is not precompact in \tilde{D} . This violates **H5**. To check the second condition, we introduce the following sets:

$$E_\varepsilon = \{x \in \text{Int}\tilde{K} \mid f(x, m_\varepsilon(x)) = 0\}, \quad 0 \leq \varepsilon \ll 1.$$

We claim that $E_\varepsilon \subseteq E_0$ for all ε small enough, which implies that E_ε is countable, by **H6**. Let us prove the claim. It is clear that E_0 consists of the x -coordinates of all equilibria of (2) in \tilde{D} . Fix $0 < \varepsilon < \varepsilon_0$ and pick any $x_0 \in E_\varepsilon$, $y_0 = m_\varepsilon(x_0)$. The solution $(x(t), y(t))$ to (2) with initial condition (x_0, y_0) satisfies (7) and (8) for t small enough. But

$$\frac{dx(t)}{dt} = f(x(t), y(t)) = f(x(t), m_\varepsilon(x(t))) \equiv 0,$$

so $(x(t), y(t)) = (x_0, y_0)$ for all $t \geq 0$, and therefore $x_0 \in E_0$. Applying Lemma 1 we have:

Lemma 4. *Under assumptions **H1-H6**, for each $0 < \varepsilon < \varepsilon_0$, there exists a set $\mathcal{C}_\varepsilon \subseteq \text{Int}\tilde{K}$ such that the forward trajectory of (7) for every point of \mathcal{C}_ε converges to some equilibrium, and the measure of $\text{Int}\tilde{K} \setminus \mathcal{C}_\varepsilon$ is zero.*

Until now, we have discussed the flow only when restricted to the locally invariant manifold M_ε . The next theorem, stated in the form given by [8], deals with more global behavior. In [8], the theorem is stated for $\varepsilon > 0$, but some properties also hold for $\varepsilon = 0$ ([9]). (We will apply this result again with a fixed $r > 1$.) The notation $[-\delta, \delta]$ stands for the cube $\{ (y_1, \dots, y_m) \mid -\delta \leq |y_i| \leq \delta \}$.

Fenichel's Third Theorem *Let ε_0 be as in Fenichel's First Theorem. Under assumption H1, if M_0 is normally hyperbolic relative to (6), then there exists $0 < \varepsilon_1 < \varepsilon_0$ and $\delta > 0$ such that for every $0 \leq \varepsilon < \varepsilon_1$ and $r > 0$, there is a function*

$$h_\varepsilon : \tilde{K} \times [-\delta, \delta] \rightarrow \mathbb{R}^n$$

such that the following properties hold:

1. *For each $x \in \tilde{K}$, $h_\varepsilon(x, 0) = x$.*
2. *The image of the map*

$$\begin{aligned} T_\varepsilon : \tilde{K} \times [-\delta, \delta] &\rightarrow \mathbb{R}^n \times \mathbb{R}^m \\ (x, \lambda) &\mapsto (h_\varepsilon(x, \lambda), \lambda + m_\varepsilon(h_\varepsilon(x, \lambda))) \end{aligned}$$

is defined as the stable manifold $W_\varepsilon^s(M_\varepsilon)$ of M_ε . For $p = (x, m_\varepsilon(x)) \in M_\varepsilon$, the stable fibers $W_\varepsilon^s(p)$, defined as $T_\varepsilon(\{x\} \times [-\delta, \delta])$, form a “positively invariant” family when $\varepsilon \neq 0$, in the sense that

$$W_\varepsilon^s(p) \cdot_{W_\varepsilon^s(M_\varepsilon)} t \subseteq W_\varepsilon^s(\phi_t(p)).$$

3. *“Asymptotic Phase”. There are positive constants k and α such that for any $p, q \in \mathbb{R}^{n+m}$, if $q \in W_\varepsilon^s(p)$, $\varepsilon \neq 0$, then*

$$|\phi_t(p) - \phi_t(q)| \leq ke^{-\alpha t}$$

for all $t \geq 0$ as long as $\phi_t(p)$ and $\phi_t(q)$ stay in $W_\varepsilon^s(M_\varepsilon)$.

4. *The stable fibers are disjoint, i.e., for $q_i \in W_\varepsilon^s(p_i)$, $i = 1, 2$, either $W_\varepsilon^s(p_1) \cap W_\varepsilon^s(p_2) = \emptyset$ or $W_\varepsilon^s(p_1) = W_\varepsilon^s(p_2)$.*
5. *The function $h_\varepsilon(x, \lambda)$ is C^r jointly in ε , x and λ . When $\varepsilon = 0$, $h_{0,\delta}(x, \lambda) = x$.*

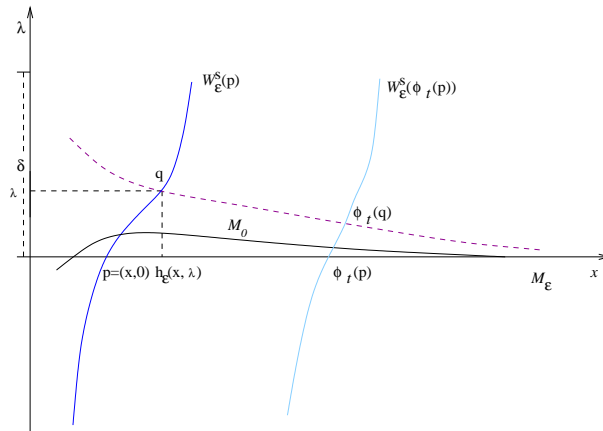


Fig. 2. To illustrate the geometric meaning of Fenichel’s Third Theorem, we sketch the locally invariant manifold and stable fibers of a system, in the case of $n=m=1$. The dimensions of the manifolds M_ε , M_0 , and stable fibers are one. M_ε is the graph of $\lambda = 0$, and M_0 is the graph of $m_0(x) - m_\varepsilon(x)$ (black curve). These manifolds may intersect at some equilibrium points. Through each point $p \in M_\varepsilon$ (x -axis), there is a stable fiber $W_\varepsilon^s(p)$ (blue curve). We call p the “base point” of the fiber. The fiber consists of the pairs $(h_\varepsilon(x, \lambda), \lambda)$, where $|\lambda| \leq \delta$. If a solution (purple dashed curve) starts on fiber $W_\varepsilon^s(p)$, after a small time t , it evolves to a point on another stable fiber $W_\varepsilon^s(\phi_t(p))$ (light blue curve); this is the “positive invariance” property.

The next lemma gives a sufficient condition to guarantee that a point is on some fiber.

Lemma 5. *Let ε_1 and δ be as in Fenichel's Third Theorem. There exists $0 < \varepsilon_2 < \varepsilon_1$, such that for every $0 < \varepsilon < \varepsilon_2$, the set*

$$\mathcal{A}_\delta := \{(x, y) \mid x \in K, \ |y - m_0(x)| \leq \frac{\delta}{2}\}$$

is a subset of $W_\varepsilon^s(M_\varepsilon)$.

To prove this lemma, we need the following result:

Lemma 6. *Let U and V be compact, convex subsets of \mathbb{R}^n and \mathbb{R}^m respectively. Suppose given a continuous function*

$$\begin{aligned}\phi : U \times V &\rightarrow \mathbb{R}^n \times \mathbb{R}^m \\ (x, y) &\mapsto (\phi_1(x, y), \phi_2(x, y))\end{aligned}$$

satisfying $\|\phi_1(x, y) - x\| \leq \rho_1$, $\|\phi_2(x, y) - y\| \leq \rho_2$ for some $\rho_1 > 0$, $\rho_2 > 0$ and all $(x, y) \in U \times V$. Then every point $(\alpha, \beta) \in U \times V$ with $\text{dist}(\alpha, \partial U) \geq \rho_1$ and $\text{dist}(\beta, \partial V) \geq \rho_2$ is in the image of ϕ .

Proof. For such a point (α, β) , consider the map $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y)) := (x, y) - (\phi_1(x, y), \phi_2(x, y)) + (\alpha, \beta)$. Thus Φ maps $U \times V$ into itself. If not, say $\Phi_1(x, y)$ is not in U , that is, $x - \phi_1(x, y) + \alpha$ is not in U . Since $\|x - \phi_1(x, y)\| \leq \rho_1$, then $\text{dist}(\alpha, \partial U) < \rho_1$, contradiction. The case when $\Phi_2(x, y)$ is not in V follows similarly. Since Φ maps $U \times V$ into itself, and the product of convex sets is still convex, by Brouwer's Fixed Point Theorem, there is some $(\bar{x}, \bar{y}) \in U \times V$ so that $\Phi(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$, which means that $(\phi_1(\bar{x}, \bar{y}), \phi_2(\bar{x}, \bar{y})) = (\alpha, \beta)$, as we wanted to prove. \square

Proof of Lemma 5. Define for each $0 \leq \varepsilon < \varepsilon_1$, the map

$$\begin{aligned}\phi_\varepsilon : \tilde{K} \times [-\delta, \delta] &\rightarrow \mathbb{R}^n \times \mathbb{R}^m \\ (x, \lambda) &\mapsto (h_\varepsilon(x, \lambda), \lambda + m_\varepsilon(h_\varepsilon(x, \lambda)) - m_0(h_\varepsilon(x, \lambda))).\end{aligned}$$

By property 5 in Fenichel's Third Theorem and the compactness of \tilde{K} , we have

$$\|h_\varepsilon((x, m_\varepsilon(x)), \lambda) - x\| \leq C_1(\varepsilon),$$

and

$$\|[\lambda + m_\varepsilon(h_\varepsilon(x, \lambda)) - m_0(h_\varepsilon(x, \lambda))] - \lambda\| \leq C_2(\varepsilon)$$

for some positive functions C_i of ε such that $C_i \rightarrow 0$ as $\varepsilon \rightarrow 0$, $i = 1, 2$. For any such ε , apply Lemma 6 with $U = \tilde{K}$, $V = [-\delta, \delta]$, $\rho_1 = C_1(\varepsilon)$, $\rho_2 = C_2(\varepsilon)$ and $\phi = \phi_\varepsilon$. Since $\text{dist}(K, \partial \tilde{K})$ and $\frac{\delta}{2}$ are independent of ε , we can pick $0 < \varepsilon_2 < \varepsilon_1$ such that $\text{dist}(K, \partial \tilde{K}) > C_1(\varepsilon)$ and $\frac{\delta}{2} > C_2(\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_2)$. By Lemma 6,

$$K \times [-\frac{\delta}{2}, \frac{\delta}{2}] \subset \phi_\varepsilon(\tilde{K} \times [-\delta, \delta]) \quad (9)$$

Define a map

$$\begin{aligned}\pi_0 : \mathbb{R}^n \times \mathbb{R}^m &\rightarrow \mathbb{R}^n \times \mathbb{R}^m \\ (x, \lambda) &\mapsto (x, \lambda + m_0(x))\end{aligned}$$

Consider the composition of π_0 and ϕ_ε : By property 2 in Fenichel's Third Theorem, its image, $\pi_0 \circ \phi_\varepsilon(\tilde{K} \times [-\delta, \delta])$, is the stable manifold $W_\varepsilon^s(M_\varepsilon)$ of M_ε . According to (9), $\pi_0(K \times [-\frac{\delta}{2}, \frac{\delta}{2}]) \subset W_\varepsilon^s(M_\varepsilon)$. Notice that $\mathcal{A}_\delta = \pi_0(K \times [-\frac{\delta}{2}, \frac{\delta}{2}])$, we are done. \square

Lemma 7. *Let ε_2 be as in Lemma 5, and δ as in Fenichel's Third Theorem. Under assumption **H7**, there exists $0 < \varepsilon_3 < \varepsilon_2$ such that for each $0 < \varepsilon < \varepsilon_3$, if $p \in D$, then there exists $T_0 \geq 0$, and $\phi_t(p) \in \mathcal{A}_\delta$ for all $t \geq T_0$.*

Proof. Setting $z = y - m_0(x)$ and $\tau = \frac{t}{\varepsilon}$, (2) becomes

$$\begin{aligned} \frac{dx}{d\tau} &= \varepsilon f(x, z + m_0(x)) \\ \frac{dz}{d\tau} &= Az - \varepsilon m'_0(x) f(x, z + m_0(x)). \end{aligned}$$

So

$$z(\tau) = z(0)e^{A\tau} - \varepsilon \int_0^\tau e^{A(\tau-s)} m'_0(x) f(x, z + m_0(x)) ds.$$

Notice that $\|e^{At}\| \leq Ce^{\beta t}$, for some positive constant C and negative β , which is greater than the real part of all eigenvalues of A . So,

$$\left\| \varepsilon \int_0^\tau e^{A(\tau-s)} m'_0(x) f(x, z + m_0(x)) ds \right\| \leq \frac{2\varepsilon MC}{|\beta|},$$

where M is an upper bound of the function $\|m'_0(x)f(x, y)\|$ on \overline{D} . Let

$$\varepsilon = \frac{\delta|\beta|}{8MC} \quad \text{and} \quad T'_0 = \max\left\{\frac{1}{|\beta|} \ln \frac{4C\|z(0)\|}{\delta}, 0\right\}.$$

Then, we have $\|z(\tau)\| \leq \frac{\delta}{2}$ for all $\tau \geq T'_0$. Back to the slow time scale, we let $T_0 = \varepsilon T'_0$. Therefore, $\phi_t(p) \in \mathcal{A}_\delta$ for all $t \geq T_0$, derived from **H7**.

Remark: Except for the normal hyperbolicity assumption, Lemma 7 is the only place where the special structure (2) was used. Consider a more general system as in (1), and assume that $g(x, m_0(x)) = 0$ on \tilde{K} for some smooth function m_0 . By the same change of variables as in the above proof, (1) is equivalent to

$$\begin{aligned}\frac{dx}{d\tau} &= \varepsilon f(x, z + m_0(x)) \\ \frac{dz}{d\tau} &= g(x, z + m_0(x)) - \varepsilon m'_0(x) f(x, z + m_0(x)).\end{aligned}$$

The only property that we need in the lemma is that for any initial condition $(x(0), z(0))$, the solution $(x(t), z(t))$ satisfies

$$\limsup_{t \rightarrow \infty} |z(t)| \leq \gamma \left(\limsup_{t \rightarrow \infty} d(t) \right)$$

where γ is a function of class \mathcal{K} , that is to say, a continuous function $[0, \infty) \rightarrow [0, \infty)$ with $\gamma(0) = 0$, and $d(t) = \varepsilon m'_0(x(t)) f(x(t), z(t) + m_0(x(t)))$. In terms of the functions m_0 and g , we may introduce the control system $dz/dt = G(d(t), z) + u(t)$, where d is a compact-valued “disturbance” function and u is an input, and $G(d, z) = g(d, z + m_0(d))$. Then, the property of input-to-state stability with input u (uniformly on d), which can be characterized in several different manners, including by means of Lyapunov functions, provides the desired condition. \square

Lemma 7 proves that every trajectory in D is attracted to A_δ and therefore is also attracted to M_ε . This will lead to our proof of the main theorem.

Proof of the main theorem.

Choose $\varepsilon^* = \varepsilon_3$, defined in Lemma 7. For any $p \in D$, there are three cases:

1. $p \in M_\varepsilon$. By Lemma 4, the forward trajectory converges to an equilibrium except for a set of measure zero.
2. $p \in \mathcal{A}_\delta \subset W_\varepsilon^s(M_\varepsilon)$. Then p is on some fiber, say $W_\varepsilon^s(\bar{p})$, where $\bar{p} = (\bar{x}, m_\varepsilon(\bar{x})) \in M_\varepsilon$. If \bar{x} is in \mathcal{C}_ε (defined in Lemma 4), then $\phi_t(\bar{p}) \rightarrow q$, for some $q \in E_0$. By the “asymptotic phase” property of Fenichel’s Third Theorem, $\phi_t(p)$ also converges to q . To deal with the case when $\bar{x} \notin \mathcal{C}_\varepsilon$, it is enough to show that the set

$$\mathcal{B}_\varepsilon = \bigcup_{\bar{x} \in \text{Int} \tilde{K} \setminus \mathcal{C}_\varepsilon} W_\varepsilon^s(\bar{p})$$

as a subset of \mathbb{R}^{m+n} has measure zero. Define

$$\mathcal{F}_\varepsilon = \left(\text{Int} \tilde{K} \setminus \mathcal{C}_\varepsilon \right) \times [-\delta, \delta].$$

Since $\text{Int}\tilde{K} \setminus \mathcal{C}_\varepsilon$ has measure zero in \mathbb{R}^n , also \mathcal{F}_ε has measure zero. On the other hand $T_\varepsilon(\mathcal{F}_\varepsilon) = \mathcal{B}_\varepsilon$, and Lipschitz maps send measure zero sets to measure zero sets, we are done.

3. $p \in D \setminus \mathcal{A}_\delta$. By Lemma 7, $\phi_t(p) \in \mathcal{A}_\delta$ for all $t \geq T_0$. Without loss of generality, we assume that T_0 is an integer. If $\phi_{T_0}(p) \in \mathcal{A}_\delta \setminus \mathcal{B}_\varepsilon$, then $\phi_t(p)$ converges to an equilibrium. Otherwise, $p \in \bigcup_{k \geq 0, k \in \mathbb{Z}} \phi_{-k}(\mathcal{B}_\varepsilon)$. Since the set \mathcal{B}_ε has measure zero and ϕ_{-k} is Lipschitz, $\phi_{-k}(\mathcal{B}_\varepsilon)$ has measure zero for all k , and the countable union of them still has measure zero. \square

5 An Example

Consider the following system:

$$\begin{aligned} \frac{dx_i}{dt} &= \gamma_i(y_1, \dots, y_m) - \beta_i(x_1, \dots, x_n), \quad i = 1, \dots, n, \\ \varepsilon \frac{dy_j}{dt} &= -d_j y_j - \alpha_j(x_1, \dots, x_n), \quad d_j > 0, \quad j = 1, \dots, m, \end{aligned} \quad (10)$$

where α_j , β_i and γ_i are smooth functions. We assume that

1. The reduced system

$$\frac{dx_i}{dt} = \gamma_i\left(-\frac{\alpha_1}{d_1}, \dots, -\frac{\alpha_m}{d_m}\right) - \beta_i(x_1, \dots, x_n) := F_i(x_1, \dots, x_n), \quad i = 1, \dots, n$$

has partial derivatives that satisfy:

$$\frac{\partial F_i}{\partial x_k} = \sum_{l=1}^m -\frac{1}{d_i} \frac{\partial \gamma_i}{\partial y_l} \frac{\partial \alpha_l}{\partial x_k} - \frac{\partial \beta_i}{\partial x_k} > 0 \text{ for } i \neq k. \quad (11)$$

2. For each i ,

$$\lim_{u \rightarrow +\infty} \min_{x \in S_i(u)} \beta_i(x) = +\infty \quad (12)$$

and

$$\lim_{u \rightarrow -\infty} \max_{x \in S_i(u)} \beta_i(x) = -\infty \quad (13)$$

where $S_i(u)$ is the set of vectors in \mathbb{R}^n whose i th coordinate is u . (For $n = 1$, this means simply that $\lim_{x \rightarrow \pm\infty} \beta_i(x) = \pm\infty$.)

3. There exists a positive constant M_j such that $|\alpha_j(x)| \leq M_j$ for all $x \in \mathbb{R}^n$.

4. The number of roots of the system of equations

$$\gamma_i(\alpha_1(x), \dots, \alpha_m(x)) = \beta_i(x), \quad i = 1, \dots, m$$

is countable.

We are going to show that on any large enough region, and provided that ε is sufficiently small, almost every trajectory converges to an equilibrium. To emphasize the need for small ε , we also show that when $\varepsilon > 1$, a limit cycle could appear.

To apply our main theorem, we take

$$L = \{ y \in \mathbb{R}^m \mid |y_j| < b_j, \quad j = 1, \dots, m \},$$

where b_j is an arbitrary positive number greater than $\frac{M_j}{d_j}$. Picking such b_j assures $y_j \frac{dy_j}{dt} < 0$ for all $x \in \mathbb{R}$ and $|y_j| = b_j$, i.e. the vector field points transversally inside on the boundary of L . Let

$$K = \{ x \in \mathbb{R}^n \mid -a_{i,2} \leq x_i \leq a_{i,1}, \quad i = 1, \dots, n \}$$

where $a_{i,1}$ and $a_{i,2}$ can be any positive numbers such that

$$\beta_i(x) > N_i := \max_{|y_j| \leq b_j} |\gamma_i(y_1, \dots, y_m)|$$

whenever $x \in \mathbb{R}^n$ satisfies that its i th coordinate $x_i \geq a_{i,1}$, and

$$\beta_i(x) < -N_i$$

whenever $x \in \mathbb{R}^n$ satisfies that its i th coordinate $x_i \leq -a_{i,2}$. All large enough $a_{i,j}$'s satisfy this condition, because of the assumption made on β .

So, we have $x_i \frac{dx_i}{dt} < 0$ for all $y \in L$, $x_i = a_{i,1}$ and $x_i = -a_{i,2}$. We then take

$$\tilde{K} = \{ x \in \mathbb{R}^n \mid -a_{i,2} - 1 \leq x_i \leq a_{i,1} + 1, \quad i = 1, \dots, n \},$$

$D = K \times L$ and $\tilde{D} = \text{Int} \tilde{K} \times L$. Thus, the vector field will point into the interior of D and \tilde{D} . Hypotheses **H5** and **H7** follow directly from this fact. (Sketch: **H7** is obvious. Suppose **H5** does not hold. Then, there exists some solution $(x(t), y(t))$ of (10) in D , and a sequence $\{t_j\}$ such that $(x(t_j), y(t_j)) \rightarrow (\bar{x}, \bar{y})$ as $j \rightarrow \infty$. Suppose that $\bar{y}_k = b_k$ for some $k \in \{1, \dots, m\}$, and $\{y_k(t_j)\}$ is strictly increasing to b_k . This will contradict the fact that $\frac{dy_i}{dt} < 0$ above the

y_k -nullcline. The other cases follow similarly.) **H3** follows from our assumption 1, and it is easy to see the other hypotheses also hold. By our main theorem, for sufficiently small ε , the forward trajectory of (10) starting from almost every point in D converges to some equilibrium.

On the other hand, convergence does not hold for large ε . Let

$$n = 1, \beta_1(x_1) = \frac{x_1^3}{3} - x_1, m = 1, \alpha_1(x_1) = 4 \tanh x_1, \gamma(y_1) = y_1, d_1 = 1.$$

It is easy to verify that $(0, 0)$ is the only equilibrium. When $\varepsilon > 1$, the trace of the Jacobian at $(0, 0)$ is $1 - \frac{1}{\varepsilon} > 0$, its determinant is $\frac{15}{\varepsilon} > 0$, so the (only) equilibrium in D is repelling. By the Poincaré-Bendixson Theorem, there exists a limit cycle in D .

Remark: The conditions (11), (12), and (13) are satisfied, in particular, if one assumes the following easier to check conditions on the functions β_i 's, α_l 's, and γ_i 's. The functions β_i are asked to be so that:

$$\frac{\partial \beta_i}{\partial x_k}(x) < 0$$

for every $i, k = 1, \dots, n$ such that $i \neq k$ (cooperativity condition among x_i variables), and also so that:

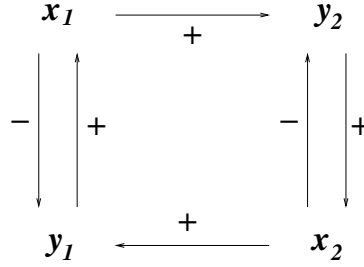
$$\lim_{x_1 \rightarrow +\infty, \dots, x_n \rightarrow +\infty} \beta_i(x_1, \dots, x_n) = +\infty \quad (14)$$

$$\lim_{x_1 \rightarrow -\infty, \dots, x_n \rightarrow -\infty} \beta_i(x_1, \dots, x_n) = -\infty. \quad (15)$$

These last conditions are very natural. They are satisfied, for example, if there is a linear decay term $-\lambda_i x_i$ in the differential equation for each x_i , and all other variables appear saturated in this rate. Since $\frac{\partial \beta_i}{\partial x_k}(x) < 0$ for all $i \neq k$, (14)-(15) imply that conditions (12) and (13) both hold. Regarding the remaining functions, we ask:

$$\sum_{l=1}^m \frac{\partial \gamma_i}{\partial y_l} \frac{\partial \alpha_l}{\partial x_k} \leq 0$$

for all $i, k = 1, \dots, n$ such that $i \neq k$. This condition can be guaranteed to hold based only upon the signs of the partial derivatives: it holds true if there is no indirect negative effect (through the variables y_l) of any variable x_k on any

**Fig. 3.** Example

other variable x_i . The diagram shown in Figure 3 illustrates one such influence graph (signs indicate signs of partial derivatives), for $n = m = 2$. Observe that this example cannot describe a monotone system (with respect to any orthant cone, i.e., it is not cooperative under any possible change of coordinates of the type $x_i \rightarrow -x_i$ or $y_l \rightarrow -y_l$). An entirely analogous example can be done for any $n = m$, the key property being that each variable x_i “represses” its associated variables y_i and the y_l ’s “enhance” some or all other variables.

6 Appendix

Definition 5 The closed half-space $H^l \subset \mathbb{R}^l$, is defined as follows:

$$H^l = \{(x_1, x_2, \dots, x_l) \in \mathbb{R}^l \mid x_1 \geq 0\}.$$

The boundary of H^l , denoted as ∂H^l , is \mathbb{R}^{l-1} .

Definition 6 A subset $M \subseteq \mathbb{R}^N$ is called a l -dimensional C^r manifold with boundary if it satisfies:

1. There exists a countable collection of open sets $V^\alpha \subseteq \mathbb{R}^N$, $\alpha \in \mathcal{I}$, where \mathcal{I} is some countable index set, so that, with $U^\alpha \equiv V^\alpha \cap M$, one has $M = \bigcup_{\alpha \in \mathcal{I}} U^\alpha$.
2. There exists a C^r diffeomorphism x^α defined on each U^α which maps U^α to some set $W \cap H^l$ where W is some open set in \mathbb{R}^l .

The boundary of M , denoted as ∂M , is the set of points in M that are mapped to ∂H^l under x^α , for some $\alpha \in \mathcal{I}$.

A compact manifold is a manifold that is compact as a topological space. The definition implies that ∂M is a well defined C^r manifold of dimension $l - 1$, and $M \setminus \partial M$ is an l -dimensional C^r manifold; see [10] for the details.

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